

Available online at www.sciencedirect.com

Topology and its Applications 154 (2007) 1089–1096

**Topology
and its
Applications**

www.elsevier.com/locate/topol

Convergence preserving mappings on topological groups

Walden Freedman

Department of Mathematics, Humboldt State University, 1 Harpst St, Arcata, CA 95521, USA

Received 5 June 2006; received in revised form 26 October 2006; accepted 26 October 2006

Abstract

It is well known that a mapping $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is convergence preserving, that is, whenever an infinite series $\sum a_n$ converges, the series $\sum \varphi(a_n)$ converges, if and only if there exists $m \in \mathbb{R}$ such that $\varphi(x) = mx$ in some neighborhood of 0. We explore convergence preserving mappings on Hausdorff topological groups, showing in particular, that if $G \times G$ is a Fréchet group, and H has no small subgroups, then a mapping $\varphi: G \rightarrow H$ is convergence preserving if and only if there is a neighborhood of the identity in G on which φ is a sequentially continuous homomorphism.

© 2006 Elsevier B.V. All rights reserved.

MSC: primary 22A05; secondary 22A10, 54D55, 54A20

Keywords: Fréchet group; Convergence preserving; Small subgroups

1. Introduction

It is well known [5,8,11] that a mapping $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is convergence preserving, that is, whenever an infinite series $\sum a_n$ converges, the series $\sum \varphi(a_n)$ converges, if and only if there exists $m \in \mathbb{R}$ such that $\varphi(x) = mx$ in some neighborhood of 0. Stated more abstractly, considering \mathbb{R} as a topological group under addition, φ is convergence preserving if and only if the restriction of φ to some neighborhood of 0 coincides with a continuous homomorphism from \mathbb{R} to \mathbb{R} . This result extends easily to functions on normed spaces over \mathbb{R} or \mathbb{C} [5]. We further extend this result to the setting of topological groups, showing in particular that for Hausdorff topological groups G and H , if $G \times G$ is a Fréchet group, and H has no small subgroups, then an appropriate generalization of this result holds.

Throughout this article, G and H will denote Hausdorff topological groups, not necessarily Abelian, and both written multiplicatively. We write e for the identity in either group, when the context makes it clear which group is meant. Otherwise, we write e_H for the identity element of H . If $g \in G$, then $\langle g \rangle$ denotes the cyclic subgroup of G generated by g . As usual, if X is a topological space, and $x \in X$, and $x \in A \subseteq X$, we say A is a neighborhood of x if there exists an open set U such that $x \in U \subseteq A$. If X is a normed space over \mathbb{R} or \mathbb{C} , we write U_X for the open unit ball of X , that is, $U_X = \{x \in X: \|x\| < 1\}$.

E-mail address: wf6@humboldt.edu (W. Freedman).

2. Main results

Recall that a mapping $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is convergence preserving if whenever an infinite series $\sum a_n$ converges, the series $\sum \varphi(a_n)$ converges. Our first task in generalizing this notion to mappings on topological groups is finding an appropriate notion corresponding to that of convergent series in \mathbb{R} . Since our group operation is multiplication, we will be considering infinite products rather than sums. Rather than requiring the sequence of partial products of a sequence (g_n) to converge, we will only require that it be a Cauchy sequence. Such a sequence (g_n) will be called Cauchy multipliable. A mapping between topological groups will then be convergence preserving if it carries Cauchy multipliable sequences to Cauchy multipliable sequences. As in the real case, any continuous homomorphism is convergence preserving. We will explore in particular the relationship between convergence preserving mappings and sequentially continuous homomorphisms

2.1. Definition.

- (a) Let (s_n) be a sequence in G . We will say (s_n) is a *Cauchy sequence* in G if for every neighborhood U of e in G , there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $s_m^{-1}s_n \in U$. In other words, (s_n) is a Cauchy sequence relative to the left uniform structure on G . Equivalently, for every symmetric neighborhood V of e in G , there exists $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies that $s_m^{-1}s_n \in V$.
- (b) Let (g_n) be a sequence in G , and for each $n \in \mathbb{N}$, let $p_n = g_1 g_2 \cdots g_n = \prod_{j=1}^n g_j$. We will say the sequence (g_n) is *Cauchy multipliable* in G if the sequence (p_n) is a Cauchy sequence in G , as defined above. Equivalently, for every symmetric neighborhood V of e in G , there exists $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies that $g_m g_{m+1} \cdots g_n \in V$.

Let $\varphi: G \rightarrow H$. We will say that φ is:

- (c) *convergence preserving* (CP) if whenever a sequence (g_n) is Cauchy multipliable in G , the sequence $(\varphi(g_n))$ is Cauchy multipliable in H ;
- (d) a *locally sequentially continuous homomorphism* (LSCH) if φ is sequentially continuous at e , and there exists a neighborhood U of e in G such that whenever $x, y \in U$, we have $\varphi(xy) = \varphi(x)\varphi(y)$. For simplicity, in this case, we will say φ is LSCH on U ;
- (e) a *local sequential homomorphism* (LSH) if φ is sequentially continuous at $e \in G$ and whenever (x_n) and (y_n) are sequences in G such that $x_n \rightarrow e$ and $y_n \rightarrow e$, there exists $N \in \mathbb{N}$ such that $\varphi(x_n y_n) = \varphi(x_n)\varphi(y_n)$ for all $n \geq N$.

Observe that any convergent sequence is a Cauchy sequence, and if (s_n) is a Cauchy sequence in G , then any subsequence of (s_n) is again a Cauchy sequence in G . It follows that if (g_n) is Cauchy multipliable in G , and the sequence (a_n) is obtained from (g_n) by grouping terms of (g_n) , then (a_n) is also Cauchy multipliable in G . It is also easy to see that (g_n) is Cauchy multipliable in G if and only if

$$\lim_{\substack{n, m \rightarrow \infty \\ m \geq n}} g_n g_{n+1} \cdots g_m = e,$$

so in particular,

$$\lim_{n \rightarrow \infty} g_n = e.$$

It is easy to see that if φ is LSCH, then φ is LSH, while if φ is CP, then $\varphi(e) = e$. On the other hand, if G is a nondiscrete group in which all convergent sequences are eventually constant, then $\varphi: G \rightarrow H$ is LSH if and only if φ is CP if and only if $\varphi(e) = e$, but clearly, φ need not be LSCH in this case.

We begin with a basic result regarding CP functions.

2.2. Lemma. *Let $\varphi: G \rightarrow H$. If φ is CP, then φ is sequentially continuous at e , and $\varphi(e) = e$.*

Proof. Suppose φ is CP. It is easy to see that $\varphi(e) = e$, while if $g_n \rightarrow e$ in G , then the sequence

$$g_1, g_1^{-1}, g_2, g_2^{-1}, \dots$$

is clearly Cauchy multipliable in G , implying that the sequence

$$\varphi(g_1), \varphi(g_1^{-1}), \varphi(g_2), \varphi(g_2^{-1}), \dots$$

is Cauchy multipliable in H , implying that $\varphi(g_n) \rightarrow e$ in H . \square

Note that if U is a neighborhood of e and $\varphi: G \rightarrow H$ is LSCH on U , then there exists a symmetric neighborhood V of e in G such that $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in V$ and such that φ is sequentially continuous on V . For let $V \subseteq U$ be a symmetric neighborhood of e such that $V^2 \subseteq U$ and let $g \in V$. Then $e_H = \varphi(g^{-1}g) = \varphi(g^{-1})\varphi(g)$, implying that $\varphi(g^{-1}) = \varphi(g)^{-1}$. If $g_n \rightarrow g$, where $g \in V$ and $g_n \in V$ for all n , then $g^{-1}g_n \in U$ for all n , and $g^{-1}g_n \rightarrow e$, so that by the sequential continuity of φ at e , we have $\varphi(g^{-1}g_n) \rightarrow e$, and thus, $\varphi(g)^{-1}\varphi(g_n) \rightarrow e$, implying that $\varphi(g_n) \rightarrow \varphi(g)$. Thus, φ is sequentially continuous on V , as claimed.

Let X be a topological space, let $A \subseteq X$, and let $a \in A$. Recall from [9] that A is said to be a *sequential neighborhood* of a if every sequence in X that converges to a is eventually inside A , that is, if (x_n) is any sequence in X such that $x_n \rightarrow a$, then $x_n \in A$ for all n sufficiently large. The space X is a *Fréchet space* (or Fréchet–Urysohn space in [6]) if every sequential neighborhood of any point in X is a neighborhood of that point. Equivalently, for any subset $A \subseteq X$ and any $x \in \bar{A}$, there exists a sequence in A that converges to x . (Note that this definition of Fréchet space has no relation to the other usual meaning of Fréchet space, namely as a complete metrizable locally convex vector space; also, Fréchet spaces are sometimes called *closure-sequential* or *N-sequential* by other authors.) A subset $A \subseteq X$ is *sequentially open* if it is a sequential neighborhood of each of its points. The space X is *sequential* if every sequentially open subset of X is open. Clearly, every Fréchet space is sequential, but not conversely [9, Example 3].

Observe that a topological group G is a Fréchet group iff every sequential neighborhood of e is a neighborhood of e . This terminology gives us an alternative way to describe mappings that are LSH or LSCH, given below.

In the remainder of the paper, given a mapping $\varphi: G \rightarrow H$, let $\tau: G \times G \rightarrow H$ and $\psi: G \times G \rightarrow H$ be defined as follows. For any $x, y \in G$,

$$\begin{aligned}\tau(x, y) &= \varphi(xy)\varphi(y)^{-1}\varphi(x)^{-1}, \quad \text{and} \\ \psi(x, y) &= \varphi(xy)\varphi(y^{-1})\varphi(x^{-1}).\end{aligned}$$

2.3. Theorem. *Let $\varphi: G \rightarrow H$ be sequentially continuous at e , and let τ , and ψ be the mappings defined above. The following are equivalent:*

- (a) φ is LSH;
- (b) $\tau^{-1}\{e\}$ is a sequential neighborhood of (e, e) in $G \times G$;
- (c) $\psi^{-1}\{e\}$ is a sequential neighborhood of (e, e) in $G \times G$,

and the following are also equivalent:

- (i) φ is LSCH;
- (ii) $\tau^{-1}\{e\}$ is a neighborhood of (e, e) in $G \times G$;
- (iii) $\psi^{-1}\{e\}$ is a neighborhood of (e, e) in $G \times G$.

Proof. Assume φ is LSH, and let $x_n \rightarrow e$ and $y_n \rightarrow e$ in G . Then $(x_n, y_n) \rightarrow (e, e)$ in $G \times G$, and since φ is LSH, there exists $M \in \mathbb{N}$ such that $n \geq M$ implies that $\tau(x_n, y_n) = e$. Thus, (a) implies (b). Assume (b) holds, and observe that this implies $(e, e) \in \tau^{-1}(e)$, which implies that $\varphi(e) = e$. Let $x_n \rightarrow e$ in G . Then $(x_n, x_n^{-1}) \rightarrow (e, e)$, so there exists $M \in \mathbb{N}$ such that $n \geq M$ implies that $\varphi(x_n^{-1}) = \varphi(x_n)^{-1}$ for all $n \geq M$. Now, suppose $x_n \rightarrow e$ and $y_n \rightarrow e$ in G . Then $(x_n, y_n) \rightarrow (e, e)$ in $G \times G$, so by (b) and the preceding observation, there exists $M \in \mathbb{N}$ such that $n \geq M$ implies that

$$\tau(x_n, y_n) = \varphi(x_n y_n)\varphi(x_n)^{-1}\varphi(y_n)^{-1} = e = \varphi(x_n y_n)\varphi(x_n^{-1})\varphi(y_n^{-1}) = \psi(x_n, y_n).$$

This shows that (b) implies (c). Now, assume (c) holds. A similar argument to the above shows that if $x_n \rightarrow e$ in G , then $\varphi(x_n^{-1}) = \varphi(x_n)^{-1}$ for all n sufficiently large. The above argument then shows that (c) implies (b). Now, suppose (b) holds. As noted above, this implies that $\varphi(e) = e$. If $x_n \rightarrow e$ and $y_n \rightarrow e$ in G , then $\varphi(x_n y_n)\varphi(x_n)^{-1}\varphi(y_n)^{-1} = e$

for all n sufficiently large, implying that φ is LSH, as claimed. The proof of the equivalence of (i), (ii), and (iii) is along the same lines, and is left to the reader. \square

Note it follows from the proof that if we define $\rho: G \rightarrow H$ by $\rho(x) = \varphi(x)\varphi(x^{-1})$, then φ is LSH implies $\rho^{-1}\{e\}$ is a sequential neighborhood of e in G , though not conversely. Thus, if G is a Fréchet group and φ is LSH, then $\varphi(x^{-1}) = \varphi(x)^{-1}$ holds in a neighborhood of e in G .

The following properties of topological groups seem to have not been defined before.

2.4. Definition. A group G has

- (a) *property (S)* if for every sequential neighborhood A of e in G , there exists a sequential neighborhood W of e in G such that $W^2 \subseteq A$;
- (b) *property (R)* if for every sequential neighborhood B of (e, e) in $G \times G$, there exists a sequential neighborhood A of e in G such that $A \times A \subseteq B$.

The following result is stated and proved in [10] in the case of a linear topological space, however the proof carries over easily to our general setting of topological groups.

2.5. Theorem. [10, Chapter 3, Theorem 2.4, p. 54] *G is a Fréchet group if and only if G is sequential and has property (S).*

Property (R) and property (S) are connected in the following way.

2.6. Lemma. *If G has property (R), then G has property (S).*

Proof. Suppose G has property (R), and let A be a sequential neighborhood of e in G . Let $m: G \times G \rightarrow G$ be multiplication, that is, $m(x, y) = xy$ for any $(x, y) \in G \times G$. Then $m^{-1}(A)$ is a sequential neighborhood of $(e, e) \in G \times G$, so there exists a sequential neighborhood W of e in G such that $W \times W \subseteq m^{-1}(A)$. Thus, $W^2 = m(W \times W) \subseteq A$, implying that G has property (S). \square

It follows that every sequential non-Fréchet group fails to have property (S), and therefore also fails to have property (R).

We know that if φ is LSCH, then φ is LSH, and that the converse is false in general. It follows from Theorem 2.3 that if $G \times G$ is a Fréchet group, then the converse does hold. This fact is part of the following corollary.

2.7. Corollary. *Among the conditions given below, the following implications hold:*

- (a) \Rightarrow (b) \Rightarrow (c),

and if in addition, G is sequential, then (c) \Rightarrow (d).

- (a) $G \times G$ is a Fréchet group.
- (b) For any group H , and any $\varphi: G \rightarrow H$, φ is LSH if and only if φ is LSCH.
- (c) For some nontrivial group H , for any $\varphi: G \rightarrow H$, φ is LSH if and only if φ is LSCH.
- (d) G is a Fréchet group.

Proof. The implication (a) \Rightarrow (b) follows immediately from Theorem 2.3, and the implication (b) \Rightarrow (c) holds trivially. To show (c) \Rightarrow (d), assume G is sequential, and suppose that G is not a Fréchet group. It follows by Theorem 2.5 that G fails to have property (S). Let A be a sequential neighborhood of e in G that is not a neighborhood of e , and such that for any sequential neighborhood B of e in G , we have $B^2 \not\subseteq A$. Let H be any nontrivial group, and let $h \in H$ such that $h \neq e_H$. Let $\varphi: G \rightarrow H$ be defined by

$$\varphi(g) = \begin{cases} e_H, & \text{if } g \in A, \\ h, & \text{if } g \notin A. \end{cases}$$

Since A is a sequential neighborhood of e , clearly, φ is LSH, but if U is any neighborhood of e in G , then $U \cap A$ is a sequential neighborhood of A , so by hypothesis, $(U \cap A)^2 \not\subseteq A$. Hence, there exist elements $x, y \in U \cap A$ such that $xy \notin A$. Therefore, we have $x, y \in U$ such that $\varphi(xy) = h$, while $\varphi(x)\varphi(y) = e_H^2 = e_H$. Since $h \neq e_H$, it follows that φ is not LSCH. \square

Note that if in (c), the group H is assumed to be non-Boolean, then the assumption that G is sequential can be dropped. For in that case, let A be a sequential neighborhood of e that is not a neighborhood of e , and choose $h \in H$ such that $h^2 \neq e_H$. Define φ as in the proof, and let $x \in U \setminus A$. Then $\varphi(x)^2 = h^2$, while $\varphi(x^2) \neq h^2$.

2.8. Remark. It appears to be an open question whether under ZFC (that is, standard Zermelo–Frankel set theory with the Axiom of Choice) there exists a Fréchet group G such that $G \times G$ is not a Fréchet group [6, Question 6.7]. However, under some additional set-theoretic assumptions, such as CH, or $\text{MA} + \neg\text{CH}$, it can be shown that such groups exist [4, Corollary 1, p. 212], [7], and [6, Theorem 6.12], an otherwise unpublished result due to A. Shibakov. It follows that under $\text{ZFC} + \text{MA}$, there exists a Fréchet group G such that $G \times G$ is not Fréchet. It is easy to see that if G is a Fréchet group, then G has property (R) if and only if $G \times G$ is a Fréchet group. Thus, any Fréchet group G such that $G \times G$ is not Fréchet provides an example of a group that fails to have property (R), but does have property (S).

A property lying between LSCH and LSH is the following.

2.9. Definition. Let $\varphi: G \rightarrow H$. We will say φ is a *locally sequentially rectangular homomorphism (LSRH)* if φ is sequentially continuous at e , and there exists a sequential neighborhood A of e in G such that $A \times A \subseteq \tau^{-1}\{e\}$.

Clearly, for $\varphi: G \rightarrow H$, we have φ is LSCH $\Rightarrow \varphi$ is LSRH $\Rightarrow \varphi$ is LSH, while if G is a Fréchet group, and $\varphi: G \rightarrow H$, then φ is LSCH if and only if φ is LSRH.

2.10. Theorem. Let $\varphi: G \rightarrow H$. If φ is LSRH, then φ is CP.

Proof. Assume φ is LSRH, and suppose φ is not CP. Let (g_n) be a Cauchy multipliable sequence in G such that $(\varphi(g_n))$ is not Cauchy multipliable in H . Then there exists a symmetric neighborhood W of e in H , and increasing sequences (m_j) and (n_j) of positive integers such that $m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \dots$, such that for all $i \in \mathbb{N}$,

$$\varphi(g_{m_i})\varphi(g_{m_i+1}) \cdots \varphi(g_{n_i}) \notin W.$$

Now, let A be a sequential neighborhood of e in G such that $A \times A \subseteq \tau^{-1}\{e\}$, and consider the following sequence of partial products

$$\begin{aligned} &g_{m_1}, g_{m_1}g_{m_1+1}, g_{m_1}g_{m_1+1}g_{m_1+2}, \dots, g_{m_1}g_{m_1+1}g_{m_1+2} \cdots g_{n_1}, \\ &g_{m_1+1}, g_{m_1+1}g_{m_1+2}, g_{m_1+1}g_{m_1+2}g_{m_1+3}, \dots, g_{m_1+1}g_{m_1+2}g_{m_1+3} \cdots g_{n_1}, \dots, g_{n_1-1}g_{n_1}, g_{n_1}, \\ &g_{m_2}, g_{m_2}g_{m_2+1}, g_{m_2}g_{m_2+1}g_{m_2+2}, \dots, g_{m_2}g_{m_2+1}g_{m_2+2} \cdots g_{n_2}, \\ &g_{m_2+1}, g_{m_2+1}g_{m_2+2}, g_{m_2+1}g_{m_2+2}g_{m_2+3}, \dots, g_{m_2+1}g_{m_2+2}g_{m_2+3} \cdots g_{n_2}, \dots, g_{n_2-1}g_{n_2}, g_{n_2}, \dots \end{aligned}$$

Since (g_n) is Cauchy multipliable, this sequence of partial products converges to e in G , so without loss of generality, we may assume that all terms of this sequence lie in A . Since $A \times A \subseteq \tau^{-1}\{e\}$, we then have

$$\varphi(g_{m_i}g_{m_i+1} \cdots g_{n_i}) = \varphi(g_{m_i})\varphi(g_{m_i+1}) \cdots \varphi(g_{n_i}) \notin W$$

for all i , contradicting the sequential continuity of φ at e . Hence, φ is CP, as desired. \square

It follows immediately from this last result that if φ is LSCH, then φ is CP.

Observe that if a group G has property (R), then for $\varphi: G \rightarrow H$, we have φ is LSH if and only if φ is LSRH. Combining this fact with Theorem 2.10 yields the following corollary.

2.11. Corollary. Assume G has property (R), and let $\varphi: G \rightarrow H$. If φ is LSH, then φ is CP.

2.12. Example (A mapping that is LSRH but not LSCH). Let X be an infinite-dimensional Banach space over \mathbb{R} that has the Schur property (weak and norm sequential convergence coincide), and let G denote X as an additive topological group under its weak topology. Then G is not sequential, and hence is not a Fréchet group since U_X is sequentially open, but is not open. On the other hand, G does have property (R), for if W is a sequential neighborhood of $(0, 0)$ in $G \times G$, then for some $n \in \mathbb{N}$, we must have $\frac{1}{n}U_X \times \frac{1}{n}U_X \subseteq W$. Let $f \in X^*$, and define $\varphi: G \rightarrow \mathbb{R}$ by

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in U_X, \\ 1, & \text{if } \|x\| \geq 1. \end{cases}$$

Then φ is LSH, and since G has property (R), φ is LSRH, and hence is CP, but is clearly not LSCH.

Recall from [3] that a topological group is said to have *no small subgroups* if there is a neighborhood of the identity of the group containing no nontrivial subgroups. This property turns out to be of some importance, as it relates to the solution of Hilbert's fifth problem. In particular, it was shown by Gleason, Montgomery, and Zippin that a locally compact group has no small subgroups iff it is a Lie group [1–3]. As an example of a group *with* small subgroups, let X be any nontrivial Hausdorff group; for each $k \in \mathbb{N}$, let $H_k = X$, and let $H = \prod_{k=1}^{\infty} H_k$, with the product topology. For another example, if X is an infinite-dimensional normed space, then as an additive topological group under its weak topology, X has small subgroups.

2.13. Example (A mapping that is CP, but not LSH). Let $G = \mathbb{R}$, and for each $k \in \mathbb{N}$, let $H_k = \mathbb{Z}_2$, and let $H = \prod_{k=1}^{\infty} H_k$, with the product topology. As noted above, H has small subgroups. For each $n \in \mathbb{N}$, let e_n denote the sequence $(0, 0, \dots, 0, 1, 0, \dots) \in H$, with a 1 in the n th component, and zeros in all others, and let

$$J_n = \{x \in \mathbb{R} \mid 2^{-n} < |x| \leq 2^{1-n}\},$$

with $J_0 = \{x \in \mathbb{R} \mid 1 < |x|\}$. Define $\varphi: G \rightarrow H$ by

$$\varphi(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or } x \in J_0, \\ e_n, & \text{if } x \in J_n, \text{ for } n \geq 1. \end{cases}$$

It is easy to see that φ is CP, but not LSH: consider the sequences $x_n = 2^{-n} = y_n$.

We now show, by modifying the proof in [8], that if we restrict H to have no small subgroups, then every CP mapping is LSH.

2.14. Theorem. Assume H has no small subgroups, and let $\varphi: G \rightarrow H$. If φ is CP, then φ is LSH.

Proof. Assume H has no small subgroups, and suppose $\varphi: G \rightarrow H$ is CP. We know from Lemma 2.2 that φ is sequentially continuous at $e \in G$, and $\varphi(e) = e$. By Theorem 2.3, it suffices to show that $\psi^{-1}\{e\}$ is a sequential neighborhood of $(e, e) \in G \times G$. Assume by way of contradiction that there exist sequences (x_n) and (y_n) in G such that $x_n \rightarrow e$ and $y_n \rightarrow e$ in G , but such that $(x_n, y_n) \notin \psi^{-1}\{e\}$ for all $n \in \mathbb{N}$. Since H has no small subgroups, let W be a symmetric neighborhood of e in H that contains no nontrivial subgroups. For each n , let k_n be a positive integer such that $\psi(x_n, y_n)^{k_n} \notin W$. Clearly, the sequence

$$x_1 y_1, y_1^{-1}, x_1^{-1}, x_1 y_1, y_1^{-1}, x_1^{-1}, \dots, x_1 y_1, y_1^{-1}, x_1^{-1}, \\ x_2 y_2, y_2^{-1}, x_2^{-1}, x_2 y_2, y_2^{-1}, x_2^{-1}, \dots, x_2 y_2, y_2^{-1}, x_2^{-1}, \dots$$

where for each $n \in \mathbb{N}$, the terms $x_n y_n, y_n^{-1}, x_n^{-1}$ are repeated k_n times, is Cauchy multipliable since its partial products converge to e . Since φ is CP, it follows that the sequence

$$\varphi(x_1 y_1), \varphi(y_1^{-1}), \varphi(x_1^{-1}), \varphi(x_1 y_1), \varphi(y_1^{-1}), \varphi(x_1^{-1}), \dots, \varphi(x_1 y_1), \varphi(y_1^{-1}), \varphi(x_1^{-1}), \\ \varphi(x_2 y_2), \varphi(y_2^{-1}), \varphi(x_2^{-1}), \varphi(x_2 y_2), \varphi(y_2^{-1}), \varphi(x_2^{-1}), \dots, \varphi(x_2 y_2), \varphi(y_2^{-1}), \varphi(x_2^{-1}), \dots$$

is Cauchy multipliable in H . Grouping terms, it follows that the sequence

$$\psi(x_1, y_1)^{k_1}, \psi(x_2, y_2)^{k_2}, \psi(x_3, y_3)^{k_3}, \dots$$

is Cauchy multipliable in H , and hence, that $\psi(x_n, y_n)^{k_n} \rightarrow e$ in H . This contradicts the fact that $\psi(x_n, y_n)^{k_n} \notin W$ for all $n \in \mathbb{N}$. Thus, φ is CP implies that φ is LSH, as desired. \square

Thus, if H has no small subgroups and $\varphi: G \rightarrow H$, then φ is LSCH $\Rightarrow \varphi$ is LSRH $\Rightarrow \varphi$ is CP $\Rightarrow \varphi$ is LSH. Combining the previous theorem with Corollary 2.11 yields the following corollary.

2.15. Corollary. Assume G has property (R), and H has no small subgroups. Let $\varphi: G \rightarrow H$. Then φ is CP if and only if φ is LSH if and only if φ is LSRH.

As demonstrated in [8] for the case $G = H = \mathbb{R}$, one can use the proof of Theorem 2.14 to produce certain pathological sequences in groups, for example, a sequence (A_j) of invertible 2×2 matrices over \mathbb{R} such that the sequence $(A_1 A_2 \cdots A_n)_{n=1}^\infty$ of partial products converges, but such that the sequence $(A_1^{-1} A_2^{-1} \cdots A_n^{-1})_{n=1}^\infty$ of partial products is divergent, or similarly such that the sequence $(A_1^2 A_2^2 \cdots A_n^2)_{n=1}^\infty$ of partial products is divergent. It follows more generally that if G is a non-Abelian Lie group, then there exists a sequence (g_n) in G that is Cauchy multipliable, but such that (g_n^{-1}) is not Cauchy multipliable, and similarly, such that (g_n^2) is not Cauchy multipliable.

2.16. Corollary. Assume H is Abelian and metrizable. The following are equivalent.

- (a) H has no small subgroups.
- (b) If G is any group, and $\varphi: G \rightarrow H$, then φ is CP implies φ is LSH.
- (c) If $\varphi: \mathbb{R} \rightarrow H$, then φ is CP implies φ is LSH.

Proof. By the previous theorem, we have (a) \Rightarrow (b), and (b) \Rightarrow (c) holds trivially, so we need only show (c) \Rightarrow (a). Suppose that H is metrizable, and has small subgroups. Let d be a left-invariant metric on H inducing the topology of H . Set

$$W_1 = \{h \in H \mid d(h, e) < 1\},$$

and let $h_1 \in W_1 \setminus \{e\}$ such that $\langle h_1 \rangle \subseteq W_1$. Let

$$W_2 = \{h \in H \mid d(h, e) < \min\{2^{-1}, d(h_1, e)\}\},$$

and let $h_2 \in W_2 \setminus \{e\}$ such that $\langle h_2 \rangle \subseteq W_2$. Inductively, we have a sequence (h_n) in $H \setminus \{e\}$ and neighborhoods W_n of e in H such that for each n , we have $\langle h_n \rangle \subseteq W_n$, such that for every $k \in \mathbb{N}$, and every $n \in \mathbb{N}$, we have $d(h_n^k, e) < 2^{1-n}$.

Let J_n be as in Example 2.13, and define $\varphi: \mathbb{R} \rightarrow H$ by

$$\varphi(x) = \begin{cases} e, & \text{if } x = 0 \text{ or } x \in J_0, \\ h_n, & \text{if } x \in J_n, \text{ for } n \geq 1. \end{cases}$$

As in Example 2.13, φ is not LSH for if $x_n = 2^{-n} = y_n$, then $x_n, y_n \rightarrow 0$, and for any n , we have $\varphi(x_n)\varphi(y_n) = (\varphi(2^{-n}))^2 = h_{n+1}^2 \in W_{n+1}$, while $\varphi(x_n + y_n) = \varphi(2^{1-n}) = h_n \notin W_{n+1}$. On the other hand, φ is CP, for suppose $\sum x_n$ converges in \mathbb{R} . Let U be any neighborhood of e in H , and choose $\varepsilon > 0$ such that $\{h \in H \mid d(h, e) < \varepsilon\} \subseteq U$. Choose $N \in \mathbb{N}$ such that for all $n \geq m \geq N$ we have $\sum_{j=m}^n 2^{1-j} < \varepsilon$, and let $K \in \mathbb{N}$ such that $i \geq K$ implies $|x_i| \leq 2^{1-N}$. Then for any $i \geq K$ and $p \geq 0$, we have

$$\begin{aligned} d(\varphi(x_i)\varphi(x_{i+1}) \cdots \varphi(x_{i+p}), e) &= d(h_{i_0} h_{i_1} \cdots h_{i_p}, e) \quad \text{for some } i_0, i_1, \dots, i_p \geq N \\ &= d(h_N^{k_N} h_{N+1}^{k_{N+1}} \cdots h_{N+r}^{k_{N+r}}, e) \quad \text{for some } r \geq 0 \text{ and } k_j \geq 0 \\ &\leq \sum_{j=N}^{N+r} d(h_j^{k_j}, e) \\ &\leq \sum_{j=N}^{N+r} 2^{1-j} \\ &< \varepsilon, \end{aligned}$$

where we use the fact that H is Abelian in the second equality. Thus, for any $i \geq K$ and $p \geq 0$, we have

$$\varphi(x_i)\varphi(x_{i+1})\cdots\varphi(x_{i+p}) \in U,$$

implying that $(\varphi(x_n))$ is Cauchy multipliable in H . Hence, φ is CP, but is not LSH. \square

As mentioned above, if G is a nondiscrete group in which all convergent sequences are eventually constant, then G has property (R), and for any H , if $\varphi: G \rightarrow H$, we have φ is LSRH if and only if φ is CP if and only if φ is LSH if and only if $\varphi(e) = e$. Thus, unless some further conditions are put on G , it is too easy for a mapping to be CP. One such condition on G is that $G \times G$ is a Fréchet group, for by Corollary 2.7, under that condition, if $\varphi: G \rightarrow H$, then φ is LSH if and only if φ is LSCH. Combining this with Corollary 2.15, we have the following corollary that generalizes the main result of [5,8,11]. In particular it implies that if G is metrizable, and H has no small subgroups, then $\varphi: G \rightarrow H$ is CP if and only if φ is LSCH.

2.17. Corollary. *Assume $G \times G$ is a Fréchet group and H has no small subgroups, and let $\varphi: G \rightarrow H$. Then φ is LSCH if and only if φ is LSRH if and only if φ is CP if and only if φ is LSH.*

We end with some open questions.

Questions.

- (a) If φ is LSH, and G fails to have property (R), must φ be CP?
- (b) Do any of the converses of the implications in Corollary 2.7 hold?
- (c) Characterize sequences (φ_n) of mappings from G to H such that for any Cauchy multipliable sequence (g_n) in G the sequence $(\varphi_n(g_n))$ is Cauchy multipliable in H .

Added in proof

Professor A. Shibakov has recently shown that every LSH mapping is CP, so the answer to question (a) above is affirmative.

Acknowledgement

The author would like to thank Professor A. Shibakov for his help in preparing this article.

References

- [1] D. Montgomery, L. Zippin, Small subgroups of finite-dimensional groups, *Ann. of Math.* 56 (2) (1952) 213–241.
- [2] D. Montgomery, L. Zippin, *Topological Transformation Groups*, Interscience, New York, 1966.
- [3] A.M. Gleason, Groups without small subgroups, *Ann. of Math.* 56 (2) (1952) 193–212.
- [4] V.I. Malyhin, D.B. Shakhmatov, Cartesian products of Fréchet topological groups and function spaces, *Acta Math. Hungar.* 60 (3–4) (1992) 207–215.
- [5] R. Rado, A theorem on infinite series, *J. London Math. Soc.* 35 (1960) 273–276.
- [6] D. Shakhmatov, Convergence in the presence of algebraic structure, in: M. Hušek, J. van Mill (Eds.), *Recent Progress in General Topology II*, Elsevier, 2002, pp. 463–484.
- [7] A. Shibakov, Countable Fréchet topological groups under CH, *Topology Appl.* 91 (1999) 119–139.
- [8] A. Smith, Convergence-preserving functions: an alternative discussion, *Amer. Math. Monthly* 98 (9) (1991) 831–833.
- [9] R.F. Snipes, Convergence and continuity: the adequacy of filterbases, the inadequacy of sequences, and spaces in which sequences suffice, *Nieuw Arch. Wisk.* 23 (1) (1975) 8–31.
- [10] A. Wilansky, *Topics in Functional Analysis*, Lecture Notes in Mathematics, vol. 45, Springer, 1967.
- [11] G. Wildenberg, Convergence-preserving functions, *Amer. Math. Monthly* 95 (6) (1988) 542–544.